Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(00)00101-5

J. Appl. Maths Mechs, Vol. 64, No. 5, pp. 723–728, 2000

0021-8928/00/$—see front matter

THE RELATIVE EQUILIBRIA OF AN ORBITAL PENDULUM SUSPENDED ON A TETHER†

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(*Received* 1 *July* 1999)

The problem of the motion of an orbital pendulum suspended on a tether is considered. The existence of relative equilibria is investigated, as well as the sufficient conditions for their stability. Particular attention is paid to the study of coupling reactions, in particular, the question of whether certain relative equilibria are not realizable because the tether does not permit compression.

The problem of the motion of an orbital pendulum, that is, a pendulum whose suspension point is fixed in a rigid body moving in a central Newtonian force field, was first formulated, it seems, by Synge [1], on the assumption that the pendulum is a point mass suspended on a weightless undeformable rod at the centre of mass of the body, Radial, tangential and normal relative equilibria of such a pendulum were found, and the necessary conditions for their stability were investigated. The same problem has been considered on the assumption that the suspension point of the pendulum may be anywhere in the body. In that formulation of the problem, again, all relative equilibria of the pendulum have been determined and the necessary conditions for their stability have been investigated. The problem of the relative equilibria of a *physical pendulum* has been investigated [3], and, in particular, all relative equilibria have been obtained, and both the necessary and sufficient conditions for their stability have been presented.

A natural generalization of such problems is to analyse the orbital dynamics of coupled systems of one or more rigid bodies. For such systems, several relative equilibria have been found analytically for two-element orbital systems, and their stability conditions have been studied. Other generalizations of the problem are possible [7].

Below, unlike to the problems previously considered, it will be assumed that the point mass is suspended from a weightless inextensible tether. We will consider the question of which of the previously found relative equilibria [2] continue to exist under the assumptions made here and we will investigate the sufficient conditions for their stability.

1. FORMULATION OF THE PROBLEM

Consider the motion of a mechanical system consisting of a rigid body of mass m_1 with centre of mass at a point G, and a point mass Q of mass m_2 connected to the body by an inextensible thread of length l. The other end of the thread is attached to the rigid body at a point P. Motion occurs under the action of forces of central Newtonian attraction with centre of attraction at a point N, which is fixed in absolute space.

Suppose the system is small compared with its distance from the attracting centre. Then we may assume with satisfactory accuracy that the motion of the centre of mass of the entire system – the point O, such that

$$(m_1 + m_2)\mathrm{NO} = m_1\mathrm{NG} + m_2\mathrm{NP}$$

and the motion of the system about the centre of mass are separable. It will henceforth be assumed that the point O is describing a circular Keplerian orbit. As usual, we introduce an orbital frame of reference $Oxyz$, in which the x axis points along an orbital tangent, the y axis is perpendicular to the orbital plane and the z axis points along the radius **NO**. Finally, as in previous treatments [2], we will assume that the rigid body does not change its orientation relative to the orbital frame of reference, while the point G may change its position relative to that frame.

†*Prikl. Mat. Mekh.* Vol. 64, No. 5, pp. 755–760, 2000.

Suppose that, in projections onto the axes of the orbital frame of reference, **GP** = (a, b, c), **PQ** = (ξ, η, ζ), **GQ** = (x, y, z). Using the resulting small parameter, we expand the potential of the forces of Newtonian gravitation in powers of that parameter up to terms of order two. Then, if ω is the orbital angular velocity of the entire system and $M = m_1m_2/(m_1 + m_2)$ is the reduced mass, the transformed potential of the system may be written as (cf. [2, 3])

$$W = M\omega^2((b+\eta)^2 - 3(c+\zeta)^2)/2 \tag{1.1}$$

The system must obey a unilateral constraint

$$f = \xi^2 + \eta^2 + \zeta^2 - l^2 \leq 0 \tag{1.2}$$

To determine relative equilibria, we use Routh's method, investigating the critical points of the function

$$W_\lambda = W + \lambda f/2 \tag{1.3}$$

The constraint will then be enforced if $\lambda \geq 0$ at the equilibrium in question. The critical points of the function (1.3) are determined by the equations

$$\begin{aligned} &\frac{\partial W_\lambda}{\partial \xi} = \lambda\xi = 0, \quad \frac{\partial W_\lambda}{\partial \eta} = M\omega^2(b+\eta) + \lambda\eta = 0, \\ &\frac{\partial W_\lambda}{\partial \zeta} = -3M\omega^2(c+\zeta) + \lambda\zeta = 0 \end{aligned} \tag{1.4}$$

together with (1.2). A solution with $\lambda > 0$ corresponds to a taut tether. This condition singles out those of the previously found solutions [2] that correspond to the conditions of the problem.

By the first equation of (1.4), either $\lambda = 0$ or $\xi = 0$. The first condition means that the tether is not taut. Then, by the second and third equations of (1.4), it can be shown that $\eta = -b$, $\zeta = -c$. Thus, the point Q lies on the x axis and its distance from the point P is at most l. In other words, for the given solutions, to a certain degree of accuracy, we may say that the points G and Q describe the same circular Keplerian orbit.

Remark. In the context of the exact formulation, without simplifying assumptions concerning approximations for the Newtonian potential, the above assertion is not quite accurate, since the finite dimensions of the bodies affect the value of the radius of the orbit in steady circular motion and these radii depend essentially on the relative orientation of the bodies. Moreover, for the same constant first integral (the value of the angular momentum) and, in some cases for the same orbital orientations, there may be not just one but several admissible orbital radii, one of which turns out to be stable while the others are unstable. In this sense, consideration of the problem of the motion of celestial bodies in the so-called satellite approximation needs further substantiation.

2. OTHER TYPES OF RELATIVE EQUILIBRIA

Let us consider solutions corresponding to $\xi = 0$. In this case, irrespective of its dependence on the value of the parameter a, the tether lies in a plane parallel to the yz plane. To simplify the geometrical interpretation, we will assume that $a = 0$. The properties of these solutions depend essentially on the values of the parameters b and c. Put $\Lambda = \lambda/(M\omega^2)$.

Let $b = 0$. Then the solutions are

$$\eta = 0, \quad \zeta = \varepsilon l, \quad \varepsilon = \pm 1, \quad \Lambda = 3(\zeta + c)/\zeta \tag{2.1}$$

When that is the case,

$$y = 0, \quad z = c + \varepsilon l$$

and the tether is parallel to the z axis. For one of these solutions, the tether is situated "above" the suspension point while for the other, it lies "below" that point. The quantity Λ will be positive if Q lies outside the segment GP. In other words, the tether cannot be taut if the point Q is situated between the suspension point and the centre of mass of the body.

Let $c = 0$. Then, by virtue of the rigid equation (1.4), there are two classes of solutions. For one of them

$$\eta = -\frac{b}{1+\Lambda}, \quad \zeta = 0; \quad y = \frac{b\Lambda}{1+\Lambda}, \quad z = 0 \tag{2.2}$$

and Λ may take any value other than -1. In this case, the tether will be parallel to the y axis or, in other words, perpendicular to the orbital plane. The factor Λ will then be positive if Q is situated between the suspension point and the orbital plane. In the other case,

$$\eta = -b/4, \quad \zeta = \varepsilon(l^2 - (b/4)^2)^{1/2}, \quad \Lambda = 3; \quad y = 3b/4, \quad z = \varepsilon(l^2 - (b/4)^2)^{1/2} \tag{2.3}$$

When this happens the point Q turns out to be at one and the same distance from the orbital plane, the distance depending on how far the suspension point is distant from the orbital plane but independent of the tether length. Such solutions exist if the tether length is at least one quarter of the distance from the suspension point to the orbital plane. For these solutions we have $\Lambda = 3$, i.e., Λ is always positive.

Finally, let $bc \neq 0$. In that case

$$\eta = -\frac{b}{\Lambda+1}, \quad \zeta = \frac{3c}{\Lambda-3}; \quad y = \frac{b\Lambda}{\Lambda+1}, \quad z = \frac{c\Lambda}{\Lambda-3} \tag{2.4}$$

Thus, the point Q is situated on a hyperbola passing through the origin and the suspension point. Draw two straight lines l_y and l_z through the suspension point, parallel to the y and z axes, respectively. Then Λ will be positive for points of the hyperbola lying within the strip between the straight line l_z and the z axis, but outside the strip between l_y and the y axis. Under these conditions, if the points Q and G coincide, there will be no reaction.

3. SUFFICIENT CONDITIONS FOR STABILITY

If the constraint is strained in steady motion, in other words, $\Lambda > 0$, one can use Routh's method to investigate the sufficient conditions for stability. To do this, it will suffice to investigate when the restriction of the second variation

$$2\delta^2 W_\lambda = M\omega^2(\Lambda\delta\xi^2 + (\Lambda+1)\delta\eta^2 + (\Lambda-3)\delta\zeta^2) \tag{3.1}$$

of the Routh function W_λ to the linear manifold

$$\delta f = \{\delta(\xi, \eta, \zeta): \ \xi\delta\xi + \eta\delta\eta + \zeta\delta\zeta = 0\} \tag{3.2}$$

is sign-definite.

For solutions (2.1), the linear manifold and the restriction of the quadratic form (3.1) to that manifold have the form

$$\delta f = \{\delta\zeta = 0\}, \quad 2\delta^2 W_\lambda = M\omega^2(\Lambda\delta\xi^2 + (\Lambda+1)\delta\eta^2)$$

For solutions with $\Lambda > 0$ this restriction is always positive-definite, the degree of instability is zero and the solution is stable in the secular sense (Fig. 1). Here and below, sets of solutions with a taut tether

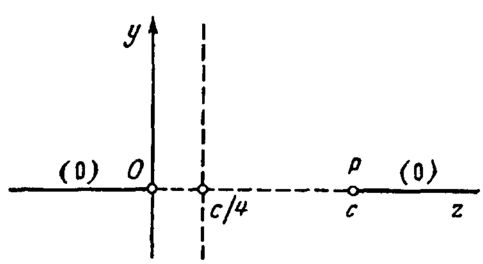

Fig. 1

will be represented in figures by a solid curve, with the degree of instability indicated in parentheses. Solutions for which the tether is not taut will be represented by a dashed curve.

For solutions (2.2), the linear manifold and the restriction of the quadratic form to the manifold have the form

$$\delta f = \{\delta\eta = 0\},\quad 2\delta^2 W_\lambda = M\omega^2(\Lambda\delta\xi^2 + (\Lambda - 3)\delta\zeta^2)$$

Then the degree of instability equals zero if $\Lambda > 3$. This is the case if the tether length is less than $b/4$. If the tether length is greater than $b/4$ but less than b, the degree of instability equals one and instability sets in. If the tether length is equal to $b/4$, the quadratic approximation does not enable is to determine whether the sufficient conditions for stability are satisfied or not. Under these conditions one has bifurcation of the given class of solutions, accompanied by the formation of solutions (2.3). For solutions (2.3), the linear manifold and the restriction of the second variation to the manifold have the form

$$\delta f = \{-b/4\delta\eta + \varepsilon(l^2 - (b/4)^2)^{1/2}\delta\zeta = 0\},\quad 2\delta^2 W_\lambda = M\omega^2(3\delta\xi^2 + 4\delta\eta^2)$$

Then the degree of instability of these solutions is zero and they are stable in the secular sense. Since these solutions bifurcate from solutions (2.2), it follows by general theorems of bifurcation theory that the solution corresponding to the point of bifurcation is also stable (Fig. 2).

Finally, for solutions (2.4), the linear manifold and the restriction of the second variation to the manifold have the form

$$\begin{aligned} &\delta f = \left\{-\frac{b}{\Lambda+1}\delta\eta + \frac{3c}{\lambda-3}\delta\zeta = 0\right\} \\ &2\delta^2 W_\lambda = M\omega^2\left(\Lambda\delta\xi^2 + \left(\frac{b^2}{(3c)^2}\frac{(\Lambda-3)^3}{(\Lambda+1)^2} + \Lambda + 1\right)\delta\eta^2\right) \end{aligned} \tag{3.3}$$

A unique real number

$$\Lambda^* = \frac{3b^{2/3} - (3c)^{2/3}}{b^{2/3} + (3c)^{2/3}}$$

exists for which the second coefficient of the quadratic form (3.3) vanishes, changing sign from positive when $\Lambda > \Lambda^*$ to negative when $\Lambda < \Lambda^*$. Thus, the degree of instability is equal to zero if

$$\Lambda > 0,\quad \Lambda > \Lambda^*$$

In that case one has secular stability. The degree of instability is equal to unity if

$$\Lambda^* > \Lambda > 0$$

In that case one has instability (Fig. 3).

The change in the type of instability in this case may be associated with the creation of a pair of steady motions. This may be demonstrated, for example, as follows. Let us substitute solutions of the third

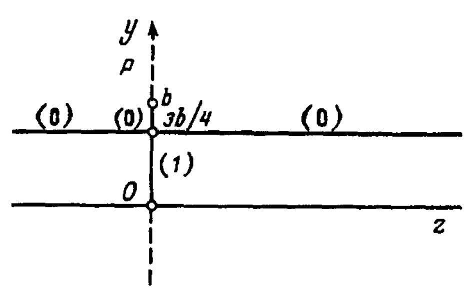

Fig. 2

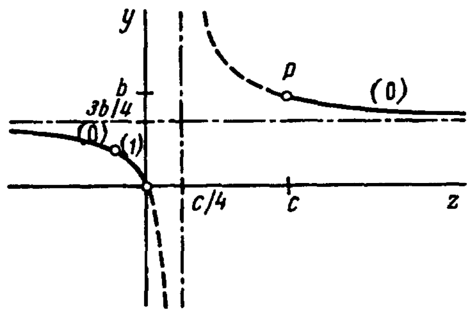

Fig. 3

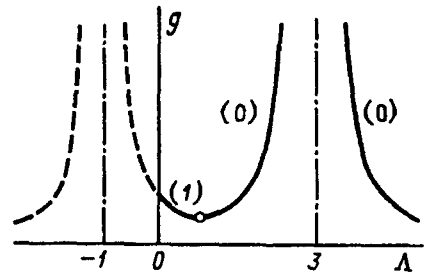

Fig. 4

type, parametrized with respect to Λ, into the equations of the constraint. We have

$$g(\Lambda) = \frac{b^2}{(\Lambda+1)^2} + \frac{(3c)^2}{(\Lambda-3)^2} = l^2 \tag{3.4}$$

The graph of the function $g(\Lambda)$ is shown in Fig. 4. The function $g(\Lambda)$ has a local minimum at the point Λ^*, with

$$g(\Lambda^*) = (b/4)^{2/3} + (3c/4)^{2/3} = g^*$$

If $l^{2/3} < g^*$, then there are only two solutions of the types considered, one of which always corresponds to negative values of the parameter Λ and may be excluded from consideration by the physical meaning of the problem.

If $l^{2/3} < g^*$, one further solution is created, which may correspond to both negative and non-negative values of the parameter $\Lambda = \Lambda^*$, whose value depends on those of the parameters b and c. Depending on the situation, the system will have two or three solutions that are meaningful in this formulation of the problem. Besides, as the investigation conducted above has shown, it is always true that only two solutions are stable.

Thus, two straight lines, separating the domains of existence of two and three physically meaningful solutions (with respect to our formulation of the problem), pass through the points of tangency of the astroid and the neighbourhood

$$g(\Lambda)/l^{2/3} = 1, \quad B^2 + C^2 = 1, \quad B = b/l, \quad C = c/l$$

which were found in [3]. This picture is symmetric about the B and C axes; accordingly, Fig. 5 illustrates only the situation in the first quadrant.

4. POSSIBLE GENERALIZATIONS

The problem of the motion of a *physical pendulum* suspended on an artificial satellite [3] may also be modified and, in the modified form, treated as a problem in the dynamics of a system constrained by a unilateral constraint. To that end, it suffices to assume that a weightless rod is attached to the body

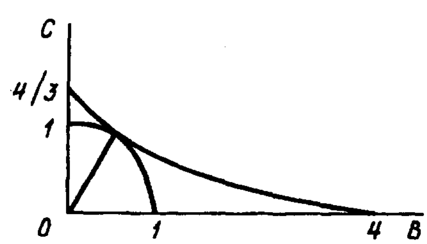

Fig. 5

by a spherical hinge, with another, axisymmetric body, whose axis of symmetry is aligned with the rod, sliding freely along the rod. If it is assumed that the suspension point of the rod and a point of the body on its axis of symmetry are connected by an inextensible tether, the resulting problem is analogous to that treated above. However, the question of whether such a system has any practical value needs further investigation. Another problem, in which it is assumed that an inextensible tether connects some point of the carrier-body and another, not necessarily axisymmetric, body, turns out to be far more complicated. Other generalizations of the problem may be associated with the consideration of aerodynamic, electromagnetic, etc. forces applied to the system [7].

We wish to thank J. Wittenburg and V. A. Sarychev for copies of their published and unpublished papers, and V. V. Beletskii for useful discussions.

This research was supported by the Jubiläumstiftung der Stadt Wien; the first author's work was also supported by the Russian Foundation for Basic Research (96-15-96051 and 99-01-00785) and the Federal Special-Purpose "Integration" Programme.

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Translated by D.L.